

Inhomogeneous Diophantine approximation of some Hurwitzian numbers

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Abstract

We continue the work of Takao Komatsu, and consider the inhomogeneous approximation constant $L(\theta, \phi)$ for Hurwitzian θ and $\phi \in \mathbb{Q}(\theta) + \mathbb{Q}$. The current work uses a compactness theorem to relate such inhomogeneous constants to the homogeneous approximation constants. Among the new results are: a characterization of such pairs θ, ϕ for which $L(\theta, \phi) = 0$, consideration of small values of $n^2 L(e^{2/s}, \phi)$ for $\phi = (r\theta + m)/n$, and the proof of a conjecture of Komatsu.

1 Introduction

The *inhomogeneous approximation constant* for a pair of real numbers θ, ϕ (with $\phi \notin \mathbb{Z}\theta + \mathbb{Z}$) is

$$L(\theta, \phi) = \liminf_{|q| \rightarrow \infty} \left\{ |q| \|q\theta - \phi\| : q \in \mathbb{Z} \right\},$$

where $\|x\|$ denotes the distance from the real number x to the nearest integer. Minkowski proved that when θ is irrational, $L(\theta, \phi) \leq 1/4$ holds for all ϕ . Grace [8] used regular simple continued fractions to construct θ with $L(\theta, 1/2) = 1/4$. Further historical details on these and related results can be found in Koksma [9]. In the middle of the twentieth century there was substantial work related to these inhomogeneous approximation constants and also to the associated inhomogeneous Markoff values. Reference [6] contains a good overview of this work and has a comprehensive list of references.

In the last decade, interest in these problems was rekindled by the authors of [3, 4, 6], and continued with articles by Christopher Pinner [19] and Takao Komatsu [10, 11, 12, 13]. In particular, Komatsu used several different types of continued fractions to compute the inhomogeneous constants when $e^{1/s}$ (for positive integer s) is paired with various ϕ in $\mathbb{Q}\theta + \mathbb{Q}$. In this article we make use of the “relative rationality” of these pairs θ, ϕ to show how the technically simpler ideas of Grace [8] and regular simple continued fractions can be used to unify and extend Komatsu’s results.

Perron [18, Section 32] defines an *arithmetic progression of order m* to be a polynomial of degree m with rational coefficients that is a function from \mathbb{N} to \mathbb{N} . The real number θ is a *Hurwitzian number of order m* if there exists a finite number of arithmetic progressions $f_1(x), \dots, f_R(x)$ of order at most m (and at least one has order m) such that

$$\theta = [b_0; b_1, \dots, b_n, f_1(1), \dots, f_R(1), f_1(2), \dots, f_R(2), \dots].$$

We use Perron’s convenient notation

$$\theta = [b_0; b_1, \dots, b_n, (f_1(i), \dots, f_R(i))_{i=1}^{\infty}].$$

Quadratic irrationals are the Hurwitzian numbers of order 0. For a nonzero integer k , $e^{2/k}$ and $\tanh(1/k)$ are examples of Hurwitzian numbers of order 1. In 1714 Roger Cotes found the continued fraction expansion of e :

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots] = [2; (1, 2j, 1)_{j=1}^{\infty}].$$

Euler (1737) proved this is indeed the continued fraction of e , and also that for integers $s \geq 2$,

$$e^{1/s} = [1; ((2j-1)s-1, 1, 1)_{j=1}^{\infty}] \tag{1}$$

and

$$\tanh(1/s) = [0; ((2j-1)s)_{j=1}^{\infty}].$$

In correspondence with Hermite, Stieljes described the continued fraction of $e^{2/k}$ for odd k :

$$e^2 = [7; (3j-1, 1, 1, 3j, 12j+6)_{j=1}^{\infty}],$$

and for integers $s \geq 1$,

$$e^{2/(2s+1)} = [1; (3(2s+1)j+s, 6(2s+1)(2j+1), 3(2s+1)j+5s+2, 1, 1)_{j=0}^{\infty}]. \quad (2)$$

For references and insight into the proofs, we refer the reader to [2, 17, 18], with an additional comment on the continued fraction of $\alpha = \coth(1/s)$. Since α is the result of applying a linear fractional transformation with integer coefficients to $\beta = e^{2/s}$, an algorithm of G. N. Raney [20] (also reported in [1]) can be used to relate the continued fractions of α and β .

Here we restrict to $\phi \in \mathbb{Q}\theta + \mathbb{Q}$ (where $\phi \notin \mathbb{Z}\theta + \mathbb{Z}$). By definition, $L(\theta, \phi_1) = L(\theta, \phi_2)$ when $\phi_1 - \phi_2 \in \mathbb{Z}\theta + \mathbb{Z}$, and so it suffices to assume that ϕ is in *reduced form*:

$$\phi = \frac{r\theta + m}{n} \quad \text{and } n \geq 2, \gcd(r, m, n) = 1 \text{ and } 0 \leq r, m < n. \quad (3)$$

The integer n will be called the *reduced denominator* of ϕ .

2 Connections with homogeneous approximation

In this section we consider θ of the form

$$\theta = [c_0; c_1 \dots, c_{n_1}, a_0, c_{n_1+1}, \dots, c_{n_1+n_2}, a_1, c_{n_1+n_2+1}, \dots], \quad (4)$$

where $\lim_{i \rightarrow \infty} a_i = \infty$, $n_j \geq 0$, and $\{c_i\}$ is a bounded sequence.

We use standard results on simple continued fractions that can be found for example in [5, 15, 18, 21]. Our principal reference is [15, Chapter 1].

Let $\theta = [b_0; b_1, b_2, \dots]$ be the simple continued fraction of the real number θ . For $i \geq 0$, $\mathcal{P}_i = (p_i, q_i)$ is called the *i*-th *convergent* of θ if $q_i > 0$ and $[b_0; b_1, b_2, \dots, b_i] = p_i/q_i$ in reduced form. Then $\mathcal{P}_0 = (b_0, 1)$, and using $\mathcal{P}_{-1} = (1, 0)$ we have

$$\mathcal{P}_{i+1} = b_{i+1}\mathcal{P}_i + \mathcal{P}_{i-1} \text{ for all } i \geq 0, \quad (5)$$

and

$$q_i | q_i \theta - p_i | = q_i \| q_i \theta \| = \mu_i^{-1}, \text{ where } \mu_i := [b_{i+1}; b_{i+2}, \dots] + [0; b_i, \dots, b_1]. \quad (6)$$

(Refer to Theorem 1 in [15, page 2] and Corollary 3 in [15, page 5].)

For θ of the form in (4), the subscripts I_j for which $b_{I_j+1} = a_j$ will be referred to as *leaping subscripts* with associated *leapers* $\mathcal{L}_j = (p_{I_j}, q_{I_j})$. The name is appropriate since from (6) the rational number given by a leaper yields a very efficient rational approximation to θ as compared with the approximations using earlier convergents. This terminology was used by Komatsu in [14] in a slightly different context.

Theorem 2.1. *Let $\phi = (r\theta + m)/n$ be in reduced form. If there exists an integer g such that*

$$g\mathcal{P}_i \equiv (m, -r) \pmod{n} \quad (7)$$

holds for infinitely many convergents of θ then

$$n^2 L(\theta, \phi) \leq g^2 \left(\limsup_{i \rightarrow \infty} \{ \mu_i : g\mathcal{P}_i \equiv (m, -r) \pmod{n} \} \right)^{-1}.$$

Moreover, if (7) holds for infinitely many leapers, then $L(\theta, \phi) = 0$.

Proof. Let $\{i_j\}$ be the infinite sequence for which $g\mathcal{P}_{i_j} \equiv (m, -r) \pmod{n}$. Then for each j there exist integers R_j, S_j such that $g\mathcal{P}_{i_j} = (m + nR_j, nS_j - r)$;

$$\begin{aligned} n^2 |(S_j - r/n)(S_j\theta - \phi - R_j)| &= |nS_j - r| |(nS_j - r)\theta - (m + R_jn)| \\ &= g^2 |q_{i_j}| |q_{i_j}\theta - p_{i_j}|; \end{aligned}$$

by (6),

$$n^2 |(S_j - r/n)(S_j\theta - \phi - R_j)| = g^2 / \mu_{i_j}. \quad (8)$$

Therefore,

$$\begin{aligned} n^2 L(\theta, \phi) &= n^2 \liminf_{|q| \rightarrow \infty} \{ |q| \|q\theta - \phi\| : q \in \mathbb{Z} \} \\ &\leq n^2 \liminf_{j \rightarrow \infty} \{ |S_j| |S_j\theta - \phi - R_j| \} \\ &= n^2 \liminf_{j \rightarrow \infty} \{ |S_j - r/n| |S_j\theta - \phi - R_j| \} \\ &= g^2 \liminf_{j \rightarrow \infty} \{ 1/\mu_{i_j} \}. \end{aligned}$$

Since (7) is a congruence modulo n , we may assume $1 \leq g < n$, giving

$$L(\theta, \phi) \leq \liminf_{j \rightarrow \infty} \{ 1/\mu_{i_j} \} \leq \liminf_{j \rightarrow \infty} \{ 1/b_{i_j} \}.$$

Therefore, $L(\theta, \phi) = 0$ when there are infinitely many leapers satisfying (7). \square

Theorem 2.1 was implicit in Grace's work [8]. We illustrate its usefulness by proving that for any integer $k \geq 3$, $L(e^{2/k}, (e^{2/k} + 1)/2) = 0$. This was proved by Komatsu for even k in [11, Theorem 3.1]. From (1) and (2), we note that the sequence of convergents for $e^{2/k}$ is completely periodic modulo 2. In fact, for odd $k = 2s + 1$, the modulo 2 sequence of convergents of $e^{2/k}$ has period

$$(1, 1), (s + 1, s), (1, 1), (0, 1), (1, 0), (1, 1), (s, s + 1), (1, 1), (0, 1), (1, 0), \quad (9)$$

where the leapers are congruent to $(1, 1), (s + 1, s), (s, s + 1)$ modulo 2. Since these are all of the congruence classes modulo 2, $L(e^{2/k}, \phi) = 0$ for all ϕ whose reduced denominator is 2. On the other hand, for even $k = 2s$ the modulo 2 period for the convergents of $\theta = e^{1/s}$ is

$$(1, 1), (s, s + 1), (s + 1, s), (1, 1), (0, 1), (1, 0), \quad (10)$$

where every leaper is congruent to $(1, 1) \pmod{2}$. Again $L(\theta, \phi) = 0$ for $\phi = (\theta + 1)/2$. But since $\lim_{i \rightarrow \infty} \mu_{6i+4} = \lim_{i \rightarrow \infty} \mu_{6i+5} = 2$, applying Theorem 2.1 with $g = 1$ gives $L(e^{1/s}, \phi) \leq 1/8$ for each of $\phi = 1/2, -\theta/2$.

Lemma 2.2. *Let $\theta = [b_0; b_1, b_2, \dots]$ be irrational and $\phi = (r\theta + m)/n$ be in reduced form. For any nonzero integer S , set*

$$\lambda(S) := \left| S - \frac{r}{n} \right| \|S\theta - \phi\|,$$

and let R be the nearest integer to $S\theta - \phi$. If $0 < n^2 \lambda(S) < 1$ then there exist integers i, g with g invertible modulo n such that either

$$(m + Rn, Sn - r) = g\mathcal{P}_i \quad \text{and} \quad n^2 \lambda(S) = \frac{g^2}{\mu_i} \quad (11)$$

or

$$b_{i+1} \neq 1 \quad \text{and} \quad n^2 \lambda(S) \geq g^2(1 - w) \quad \text{for some } 0 \leq w \leq [0; b_{i+1}]. \quad (12)$$

Moreover, if $n^2 \lambda(S) < 1/2$, then (11) must hold.

Proof. Define the integers $M := m + Rn$ and $N := Sn - r$. Then calculation gives

$$|N| |N\theta - M| = n^2 \lambda(S).$$

Since $0 < n^2 \lambda(S) < 1$, then $N \neq 0$ and M/N is a rational that satisfies

$$\left| \theta - \frac{M}{N} \right| < \frac{1}{N^2}.$$

By Theorem 10 in [15, page 16], there exist integers i, g such that either $(M, N) = g\mathcal{P}_i$ or $b_{i+1} \neq 1$ and $(M, N) = g(d\mathcal{P}_i + \mathcal{P}_{i-1})$ where d equals 1 or $b_{i+1} - 1$. In either case, $\gcd(g, n)$ must divide both M and N , and so each of r, m . The fact that ϕ is reduced therefore implies g is invertible modulo n . In addition, by Corollary 2 in [15, page 11], if $n^2\lambda(S) < 1/2$ then $(M, N) = g\mathcal{P}_i$. Also, if $(M, N) = g\mathcal{P}_i$ then (8) yields $n^2\lambda(S) = g^2/\mu_i$, which is (11). It remains to prove $(M, N) = g(d\mathcal{P}_i + \mathcal{P}_{i-1})$ implies (12).

We note that $d = b_{i+1} - 1$ gives $d\mathcal{P}_i + \mathcal{P}_{i-1} = \mathcal{P}_{i+1} - \mathcal{P}_i$, and so the two possibilities can be combined as $(M, N) = g(\mathcal{P}_j \pm \mathcal{P}_{j-1})$ for $j = i, i+1$ where the upper sign is taken when $j = i$ and the lower sign when $j = i+1$. Therefore,

$$\begin{aligned} n^2\lambda(S) &= |N| |N\theta - M| \\ &= g^2(q_j \pm q_{j-1}) |(q_j \pm q_{j-1})\theta - (p_j \pm p_{j-1})| \\ &= g^2(q_j \pm q_{j-1}) |(q_{j-1}\theta - p_{j-1}) \pm (q_j\theta - p_j)| \\ &= g^2(q_j \pm q_{j-1}) \left(\|q_{j-1}\theta\| \mp \|q_j\theta\| \right), \end{aligned}$$

since the differences $q_k\theta - p_k$ alternate in sign. Then (6) implies

$$n^2\lambda(S) = g^2 \left(1 \pm \frac{q_{j-1}}{q_j} \right) \left(\frac{q_j}{q_{j-1}\mu_{j-1}} \mp \frac{1}{\mu_j} \right). \quad (13)$$

For $x := [0; b_j \dots, b_1]$ and $y := [0; b_{j+1}, b_{j+2}, \dots]$,

$$\mu_{j-1} = \frac{1}{x} + y \quad , \quad \mu_j = \frac{1}{y} + x,$$

and $q_{j-1}/q_j = x$ by Theorem 4 in [15, page 6]. Putting these into (13) yields

$$n^2\lambda(S) = g^2(1 \pm x) \left(\frac{1}{1+xy} \mp \frac{y}{1+xy} \right) = g^2 \frac{(1 \pm x)(1 \mp y)}{1+xy}$$

where $0 \leq x \leq [0; b_j]$ and $0 < y < [0; b_{j+1}]$. When the upper sign holds (that is, when $j = i$), $x \geq 0$ and $y \leq 1$ yield $n^2\lambda(S) \geq g^2(1-y)$ and $w = y$ satisfies conclusion (12). Analogously, $w = x$ can be used for the lower sign. \square

Theorem 2.3. *Let θ be as in (4) and $\phi = (r\theta + m)/n$ be in reduced form. Then $L(\theta, \phi) = 0$ if and only if there exist infinitely many leapers \mathcal{L}_j such that $g_j \mathcal{L}_j \equiv (m, -r) \pmod{n}$ for an integer g_j that is invertible modulo n .*

Proof. Let $\{S_k\}$ be an infinite sequence of nonzero integers such that

$$L(\theta, \phi) = \lim_{k \rightarrow \infty} |S_k| \|S_k \theta - \phi\|.$$

If $L(\theta, \phi) = 0$, then

$$0 = L(\theta, \phi) = \lim_{k \rightarrow \infty} \left| S_k - \frac{r}{n} \right| \|S_k \theta - \phi\| = \lim_{k \rightarrow \infty} \lambda(S_k).$$

Restricting to k satisfying $n^2 \lambda(S_k) < 1/2$, for each such k Lemma 2.2 implies there exist i_k and invertible g_k modulo n such that (11) holds. Then

$$b_{i_k+1} + 2 \geq \mu_{i_k} = \frac{g_k^2}{n^2 \lambda(S_k)} \longrightarrow \infty.$$

The condition on $\{c_i\}$ in (4) implies i_k is a leaping subscript for sufficiently large k .

The converse was proved in Theorem 2.1 □

Corollary 2.4. *Let θ be as in (4) and $\phi = (r\theta + m)/n$ be in reduced form. If $L(\theta, \phi) = 0$ then $L(\theta, g\phi) = 0$ for every integer g that is not a multiple of n . In particular, for any $n \geq 2$ there exists m/n such that $L(\theta, m/n) = 0$ if and only if $L(\theta, m_1/n) = 0$ for all $m_1 \in \mathbb{Z}$, $m_1/n \notin \mathbb{Z}$.*

Proof. Let g be an integer that is not a multiple of n . By Theorem 2.3, $L(\theta, \phi) = 0$ implies there exist infinitely many leapers \mathcal{L}_{j_k} such that $g_k \mathcal{L}_{j_k} \equiv (m, -r) \pmod{n}$ for some invertible $g_k \pmod{n}$, and so

$$g_k g \mathcal{L}_{j_k} \equiv (gm, -gr) \pmod{n} \text{ for all } k.$$

Setting $d := \gcd(g, n)$ and $h := g/d$ this implies

$$g_k h \mathcal{L}_{j_k} \equiv (hm, -hr) \pmod{n/d} \text{ for all } k.$$

Since $g_k h$ is invertible modulo n/d , from Theorem 2.3 we obtain $L(\theta, g\phi) = 0$. □

Henceforth, we'll restrict consideration to a slight generalization of $e^{2/k}$; namely,

$$\begin{aligned} \theta = [a_0; c_1, \dots, c_{n_1}, a_1, c_{n_1+1}, \dots, c_{n_1+n_2}, a_2, \dots], \text{ where } \lim_{i \rightarrow \infty} a_i = \infty \\ \text{and either } \{c_i\} \text{ is a finite sequence or } \limsup_{i \rightarrow \infty} c_i = 1. \end{aligned} \tag{14}$$

Theorem 2.5. *Let $\theta = [b_0; b_1, b_2, \dots]$ be as in (14) and $\phi = (r\theta + m)/n$ be in reduced form. If $0 < n^2 L(\theta, \phi) < 1$, then there exist infinitely many non-leaping convergents $\mathcal{P}_i \equiv (m, -r) \pmod{n}$, and*

$$n^2 L(\theta, \phi) = \left(\limsup_{i \rightarrow \infty} \{\mu_i : \mathcal{P}_i \equiv (m, -r) \pmod{n}\} \right)^{-1}. \quad (15)$$

Proof. From (14), there exists I such that for $i \geq I$,

$$b_{i+1} \neq 1 \iff i \text{ is a leaping subscript}. \quad (16)$$

Let $\{S_j\}$ be an infinite sequence of nonzero integers such that

$$L(\theta, \phi) = \lim_{j \rightarrow \infty} \lambda(S_j).$$

From $0 < n^2 L(\theta, \phi) < 1$ it follows that $0 < n^2 \lambda(S_j) < 1$ holds for infinitely many j and Lemma 2.2 can be applied: For each such S_j , we obtain a subscript $i = i_j$ such that one of the conclusions of the lemma holds. By (16), if $\lambda(S_j)$ satisfies (12) for sufficiently large j , then the subscript i_j must be leaping. If an infinite subsequence S_j were to satisfy (12) with leaping subscript i_j and associated w_j , then

$$n^2 \lambda(S_j) \geq g_j^2 (1 - w_j) \quad \text{where } w_j \leq [0; b_{i_j+1}] \longrightarrow 0,$$

and we would obtain the contradiction

$$n^2 L(\theta, \phi) \geq \lim_{j \rightarrow \infty} g_j^2 \geq 1.$$

(This is similar to the argument in [21, p. 116].) Therefore, for sufficiently large j , $\lambda(S_j)$ satisfies (11) for some $i = i_j$ that is not leaping — else $L(\theta, \phi)$ would be zero. Since $\limsup_{i \rightarrow \infty} c_i = 1$, then $\mu_{i_j} \leq 3$ for all but finitely many j , and

$$\lim_{j \rightarrow \infty} \frac{g_j^2}{3} \leq \lim_{j \rightarrow \infty} \frac{g_j^2}{\mu_{i_j}} = n^2 L(\theta, \phi) < 1,$$

which implies $g_j = 1$ for all sufficiently large j . Therefore, $\mathcal{P}_{i_j} \equiv (m, -r) \pmod{n}$ for infinitely many non-leaping i_j and also (15) holds. \square

The hypothesis $L(\theta, \phi) > 0$ in Theorem 2.5 guarantees that at most finitely many leapers are congruent to $(m, -r) \pmod{n}$. It's worth noting that $n^2 L(\theta, \phi) < 1$ implies the existence of infinitely many convergents $\mathcal{P}_i \equiv g(m, -r) \pmod{n}$ with $g = 1$.

We return to the earlier question of calculating $L(e^{1/s}, \phi)$ for ϕ whose reduced denominator equals 2. Recall the sequence of convergents of θ is completely periodic modulo 2 with period given in (10). Since $\lim_{j \rightarrow \infty} \mu_{6j+k} = 2$ for all $k \not\equiv 0 \pmod{3}$, application of Theorem 2.5 gives $L(e^{1/s}, \phi) = 1/8$ for $\phi = 1/2, e^{1/s}/2$.

3 Komatsu's Conjecture

In Theorem 3.3 of this section we prove a generalization of the conjecture of T. Komatsu [13, p. 241] that for integers $s \geq 1$, $n^2 L(e^{1/s}, 1/n) = 0$ or $1/2$ for all $n \geq 2$.

Proposition 3.1. *Let k, n be positive integers with $n \geq 2$. For any sequence of integers $\{b_j\}$, define a sequence $\{s_j\} \subset \mathbb{Z}^k$ inductively using any initial values $s_0, s_1 \in \mathbb{Z}^k$, and*

$$s_j \equiv b_j s_{j-1} + s_{j-2} \pmod{n} \text{ for all } j \geq 2. \quad (17)$$

If $\{b_j\}$ is periodic modulo n , then $\{s_j\}$ is periodic. If $\{b_j\}$ is completely periodic modulo n , then $\{s_j\}$ is also completely periodic.

Proof. If b_{i+1}, \dots, b_{i+t} is a period for $\{b_j\}$, consider the following sequence of pairs:

$$(s_{i-1}, s_i), (s_{i-1+t}, s_{i+t}), (s_{i-1+2t}, s_{i+2t}), \dots$$

This infinite sequence eventually has a repetition modulo n . Because $\{s_j\}$ satisfies the recurrence (17) and b_{i+1}, \dots, b_{i+t} is a period for $\{b_j\}$, the first repetition in this sequence will identify the beginning of a period for $\{s_j\}$. Since s_{i-1} is determined by $s_{i-1} \equiv s_{i+1} - b_{i+1}s_i \pmod{n}$, when $\{b_j\}$ is completely periodic modulo n , $\{s_j\}$ must also be completely periodic. \square

In particular, since the partial quotient sequence of every Hurwitzian number is periodic modulo every integer $n \geq 2$, its sequence of convergents is periodic modulo n .

Henceforth, for θ of the form in (14) we further restrict to $n \geq 2$ for which the partial quotient sequence of θ is periodic modulo n . If b_{I+1}, \dots, b_{I+T} is a period for the partial quotient sequence such that $\mathcal{P}_{I+1}, \dots, \mathcal{P}_{I+T}$ is a period for the convergents, we define $M_j := \limsup_{k \rightarrow \infty} \mu_{j+kT}$ for all $j = I+1, \dots, I+T$ and observe that

$$M_j = \infty \iff \limsup_{k \rightarrow \infty} b_{j+kT+1} = \infty \iff \mathcal{P}_{j+kT} \text{ is a leaper for infinitely many } k. \quad (18)$$

Theorem 3.2. *Let θ have the form given in (14). Let $n \geq 2$ be such that the partial quotient sequence of θ is periodic modulo n , and b_{I+1}, \dots, b_{I+T} and $\mathcal{P}_{I+1}, \dots, \mathcal{P}_{I+T}$ be as set up above. If m, r are integers with $\gcd(m, r, n) = 1$ for which there exists $i > I$ with $\mathcal{P}_i \equiv (m, -r) \pmod{n}$, we set*

$$M := \max\{M_j : I+1 \leq j \leq I+T \text{ and } \mathcal{P}_j \equiv (m, -r) \pmod{n}\}.$$

If $M \neq 1$ then $n^2 L(\theta, (m+r\theta)/n) = M^{-1}$.

Proof. Set $\phi := (m+r\theta)/n$, and let j be such that $1 \leq j \leq T$, $\mathcal{P}_j \equiv (m, -r) \pmod{n}$, and $M_j = M \neq 1$. The observation in (18) combined with Theorem 2.3 gives the conclusion for $M = \infty$. We may therefore assume M is finite, and that by (18) at most finitely many $\mathcal{P}_j \equiv (m, -r) \pmod{n}$ are leapers, that in turn gives $L(\theta, \phi) > 0$. Since

$$(p_{j+kT}, q_{j+kT}) = \mathcal{P}_{j+kT} \equiv \mathcal{P}_j \equiv (m, -r) \pmod{n} \text{ for all } k \geq 0,$$

then

$$n^2 \lambda(S_k) = \frac{1}{\mu_{j+kT}} \text{ for } S_k := (q_{j+kT} + r)/n.$$

By Theorem 2.1,

$$0 < n^2 L(\theta, \phi) \leq n^2 \liminf_{k \rightarrow \infty} \lambda(S_k) = \liminf_{k \rightarrow \infty} \frac{1}{\mu_{j+kT}} = \frac{1}{M} < 1,$$

and the conclusion follows from Theorem 2.5. \square

Theorem 3.3. [A generalization of Komatsu's conjecture] *Let θ be an irrational whose continued fraction has the form given in (14), and let $n \geq 2$ be such that the partial quotient sequence of θ is completely periodic modulo n . If each $n_i \in \{0, 2\}$ then*

$$n^2 L(\theta, \phi) \in \{0, 1/2\} \text{ for both } \phi = 1/n, \phi = -\theta/n.$$

In particular, for every $k \geq 2$ and every $n \geq 2$

$$n^2 L(e^{2/k}, \phi) \in \{0, 1/2\} \text{ for both } \phi = 1/n, -e^{2/k}/n.$$

Proof. The fact that $n_i \in \{0, 2\}$ implies every M_j equals ∞ or 2. By Proposition 3.1, the sequence of convergents of θ is completely periodic modulo n . If T is a period length, then

$$\mathcal{P}_{T-1} \equiv \mathcal{P}_{-1} = (1, 0) \pmod{m} \text{ and } \mathcal{P}_{T-2} \equiv \mathcal{P}_{-2} = (0, 1) \pmod{m}.$$

Since $M \in \{\infty, 2\}$, the conclusion follows from Theorem 3.2. \square

Theorem 3.4. *Let $k \geq 1$, $n \geq 2$. If $\gcd(n, k) \neq 1$ then*

$$n^2 L(e^{2/k}, 1/n) = n^2 L(e^{2/k}, -e^{2/k}/n) = 1/2.$$

Proof. By Theorems 2.3 and 3.3 it suffices to prove that no component of any leaper of $\theta = e^{2/k}$ is divisible by n . Since $\gcd(n, k) \neq 1$ we first consider all sequences modulo k .

When $k = 2s$, the partial quotient sequence is completely periodic modulo k with period $1, s-1, 1$, and the period of the sequence of convergents is

$$(1, 1), (s, s-1), (s+1, s), (1, -1), (0, 1), (1, 0),$$

where the leapers are $(1, \pm 1) \pmod{k}$. When $k = 2s+1$, then the partial quotient sequence has period $1, s, 0, s, 1$ modulo k , and the sequence of convergents has period

$$(1, 1), (-s, s), (1, 1), (0, -1), (1, 0), (1, -1), (-s, -s), (1, -1), (0, 1), (1, 0),$$

where the leapers are either $(1, \pm 1)$ or $(-s, \pm s) \pmod{k}$. In each case we have shown that each component of every leaper is relatively prime to k , and therefore cannot be divisible by n . The conclusion follows from Theorems 2.3 and 3.3. \square

Earlier we proved $L(e^{1/s}, \phi) = 1/8$ for each of $\phi = 1/2, e^{1/s}/2$, a special case of the last result. The theorem also generalizes [13, Theorem 3], that $n^2 L(e^{1/s}, 1/n) = 1/2$ when n divides s .

4 When is $L(e^{1/s}, \phi)$ zero?

Theorem 4.1. *Let s, n be positive integers with $n \geq 2$, and let $\mathcal{L}_i = \mathcal{P}_{3i} = (P_i, Q_i)$ be the i -th leaper of $e^{1/s}$.*

(a) *Then $\{\mathcal{L}_i\}$ is a completely periodic sequence modulo n with period*

$$\mathcal{L}_0, \dots, \mathcal{L}_{K-1}, \mathcal{L}_K, \mathcal{L}_{K-1}, \dots, \mathcal{L}_0, \mathcal{L}_0^*, \dots, \mathcal{L}_{K-1}^*, \mathcal{L}_K^*, \mathcal{L}_{K-1}^*, \dots, \mathcal{L}_0^* \quad (19)$$

where $K = \lfloor n/2 \rfloor$ and $\mathcal{L}_i^ := (P_i, -Q_i)$.*

(b) *If $\gcd(n, 2s) = 1$, then (19) is a minimal period for the leapers of $e^{1/s}$ modulo n .*

(c) *For all $1 \leq s < n$, the i -th leaper of $e^{1/(n-s)}$ is $(-1)^i(Q_i, P_i) \pmod{n}$*

Proof. Perron [18, Section 31] proved that for $\theta = [a_0; c_1, c_2, a_1, \dots, a_i, c_1, c_2, a_{i+1}, \dots]$ the subsequence $\mathcal{P}_2, \mathcal{P}_5, \dots, \mathcal{P}_{3i+2}, \dots$ of convergents of θ satisfies the second-order recurrence

$$\mathcal{P}_{3i+2} = (a_i(c_1 c_2 + 1) + c_1 + c_2) \mathcal{P}_{3i-1} + \mathcal{P}_{3i-4} .$$

Therefore, the sequence of leapers of

$$e^{1/s} = [1; (2sj - (s+1), 1, 1)_{j=1}^{\infty}]$$

satisfies the recurrence

$$\mathcal{L}_{-1} = (1, -1), \mathcal{L}_0 = (1, 1), \mathcal{L}_{j+1} = A_j \mathcal{L}_j + \mathcal{L}_{j-1}, \quad (20)$$

for $k := 2s$ and $A_j := (2j+1)k$, a sequence that is completely periodic modulo n .

Since $A_K = (2K+1)k \equiv 0 \pmod{n}$, then $\mathcal{L}_{K+1} \equiv \mathcal{L}_{K-1} \pmod{n}$. Also, for all $j \geq 0$,

$$A_{K+j} + A_{K-j} = 2(2K+1)k \equiv 0 \pmod{n},$$

and an inductive argument using the generating recurrence (20) yields

$$\mathcal{L}_{K+j} \equiv \mathcal{L}_{K-j} \pmod{n} \text{ for all } j \geq 0. \quad (21)$$

In particular, for $j = K, K+1$,

$$\mathcal{L}_{n-1} \equiv \mathcal{L}_0 = (1, 1) = \mathcal{L}_{-1}^* \pmod{n}; \mathcal{L}_n \equiv \mathcal{L}_{-1} = (1, -1) = \mathcal{L}_0^* \pmod{n};$$

again using recurrence (20) inductively,

$$\mathcal{L}_{n+j} \equiv \mathcal{L}_j^* \pmod{n} \text{ for all } j.$$

In combination with (21) this implies (19) is a period for the leapers modulo n .

Further, if T is a period-length of the leapers, then

$$A_T \mathcal{L}_T + \mathcal{L}_{T-1} = \mathcal{L}_{T+1} \equiv \mathcal{L}_1 \equiv A_0 \mathcal{L}_0 + \mathcal{L}_{T-1} \pmod{n},$$

implying $(0, 0) \equiv (A_T - A_0) \mathcal{L}_0 \equiv 2Tk(1, 1) \pmod{n}$. When $\gcd(n, k) = 1$, T must be divisible by n . The fact that

$$\mathcal{L}_n \equiv \mathcal{L}_0^* \not\equiv \mathcal{L}_0 \pmod{n}$$

proves (19) is a minimal period for the leapers of $e^{1/s}$.

It remains to prove (c). For this, we define $\{\mathcal{M}_j\}$ to be the sequence $\mathcal{M}_j := (Q_j, P_j)$ where (P_j, Q_j) is the j -th leaper of $e^{1/s}$. Then $\{\mathcal{M}_j\}$ also satisfies the recurrence (20) with initial values $\mathcal{M}_{-1} = (-1, 1), \mathcal{M}_0 = (1, 1)$, and the sequence $\mathcal{N}_j := (-1)^j \mathcal{M}_j$ satisfies the recurrence

$$\mathcal{N}_{-1} = (1, -1), \mathcal{N}_0 = (1, 1), \mathcal{N}_{j+1} = -A_j \mathcal{N}_j + \mathcal{N}_{j-1}.$$

Since

$$(2j+1)2(n-s) \equiv -(2j+1)2s \equiv -A_j \pmod{n},$$

this is the recurrence for the leapers of $e^{1/(n-s)}$. \square

In 1918, D. N. Lehmer [16] investigated the modulo n period of the convergents for certain Hurwitzian numbers. More recently, C. Elsner [7] used generating functions to prove results on the period length of the modulo n sequence of leapers of e , and Takao Komatsu [10, 11, 12, 13] found the period length of the modulo n leapers of $e^{1/s}$ always divides $2n$ and it divides n when n is even. Both Elsner and Komatsu applied their results to homogeneous approximation over congruence classes.

Corollary 4.2. *Let $s \geq 1$, $n \geq 2$ be integers, and define $\theta := e^{1/s}$. If $L(\theta, (m+r\theta)/n) = 0$, then $L(\theta, (m-r\theta)/n) = 0$.*

Proof. The conclusion follows from Theorem 2.3 and the form of the period in (19). \square

Corollary 4.3. *Let $n \geq 2$ be a odd integer. Then for all $m \not\equiv 0 \pmod{n}$,*

$$L(e^{2/(n+1)}, m/n) = 0 \quad \text{and} \quad L(e^{2/(n-1)}, -m e^{2/(n-1)}/n) = 0. \quad (22)$$

Proof. Since n is odd, $s := (n+1)/2$ is an integer. The first leaper of $e^{1/s}$ can be calculated using recurrence (20) with $k = n+1$:

$$\mathcal{L}_1 = A_0(1, 1) + (1, -1) \equiv 2(1, 0) \pmod{n},$$

and from Theorem 4.1(c), the first leaper of $e^{2/n-1}$ is $-(0, 2)$. Therefore, Theorem 2.3 implies (22) for $m = 2$, and the conclusion follows from Corollary 2.4. \square

Theorem 4.4. *Let s be a positive integer. If n_1, n_2 are relatively prime integers for which $L(e^{1/s}, 1/n_1) = L(e^{1/s}, 1/n_2) = 0$ then $L(e^{1/s}, 1/(n_1 n_2)) = 0$.*

Proof. Since $L(e^{1/s}, 1/n_i) = 0$, the form of the period of the leapers of $e^{1/s}$ yields *even* $1 \leq j_i = 2r_i \leq 2n_i$ with $Q_{2r_i} \equiv 0 \pmod{n_i}$. Using the Chinese Remainder Theorem, the system $r \equiv r_i \pmod{n_i}$ has a solution $r \pmod{n_1 n_2}$, and $Q_{2r} \equiv Q_{2r_i} \equiv 0 \pmod{n_i}$ for each i . Therefore, we have found a subscript $j = 2r$ such that $Q_j \equiv 0 \pmod{n_1 n_2}$, and $L(e^{1/s}, 1/n_1 n_2) = 0$. \square

Theorem 4.5. *Let s be a positive integer and let $n \geq 3$ be odd. Then for any reduced $\phi = (m + r e^{1/s})/n$ it is possible to check whether or not $L(e^{1/s}, \phi)$ is zero in fewer than $n/2$ multiplications modulo n . In fact, if n has t distinct prime divisors, the number of operations can be reduced to $n/2^t$ multiplications modulo n .*

Proof. The form of the period in (19) allows one to conclude whether or not a leaper has the form $g(m, -r) \pmod{n}$ within $n/2$ applications of the recurrence (20). Theorem 4.4 reduces the question to checking the period modulo each prime power divisor of n . \square

The algorithm implicit in the proof of Theorem 4.5 can be used to verify that the following values should be added to the list given in [13, p. 241] of all values of s for which $L(e^{1/s}, 1/n) = 0$ (for $n \leq 49$):

n	$s \pmod{n}$
23	12
25	13, 23
29	15
43	25
47	11, 17, 33, 43
49	1, 22, 46

In particular, notice that (22) ensures all of $(n, s) = (23, 12), (25, 13), (29, 15)$ must be included in the table.

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